Fullness on Grassmannians via Combinatorial Constructions
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Fullness on grassmannians

Notation
- $K = \mathbb{K}$, char $K = 0$
- $X = \text{Gr}(k, n)$

Theorem [K]
- $D^b(\text{Coh } X) = (S^1U)$, where:
  - $U$ tautological bundle,
  - $\lambda$ all Young diagrams that fit in rectangle $k \times (n-k)$.

Proof [K]
- $K \cong O_3 \in D^b(\text{Coh } X \times X)$ Koszul complex
- $F \cong p_2^*(K \otimes p_1^*F) \in D^b(\text{Coh } X)$ written in terms of Schur functors of $U$

Combinatorial constructions [HLS]

Notation
- Reductive group $G$
- Maximal torus $T \subset G$
- Weyl group $W$
- $\rho = \frac{1}{2} \sum$ (positive roots)
- $G$-representation $R$
- $\beta_i$ are $T$-weights of $R$
- $M = \text{Hom}(T, G_m)$ character lattice
- Dominant chamber $M_{\mathbb{R}}$

$R$ quasi-symmetric

For any line $L \subset M$, the sum of $\beta_i \in L$ must be 0

Zonotope $\Sigma \subset M_{\mathbb{R}}$
- $\Sigma = \text{Minkowski sum of } [0, \beta]$

Polytope $\nabla \subset M_{\mathbb{R}}$
- (Roughly) defined by properties:
  - $W$-invariant
  - $M_{\mathbb{R}} \cap \nabla = M_{\mathbb{R}} \cap (-\rho + \frac{1}{2} \Sigma)$

Category $\mathcal{M}(\delta + \nabla) \subset D^b(R/G)$
- $\delta \in M_{\mathbb{R}}$
- $D^b(\text{Coh } R/G) \cong D^b_G(\text{Coh } R)$
- Character of $U \in \text{Rep } G$ is contained in $\delta + \nabla$
- $\mathcal{M}(\delta + \nabla)$ is split generated by these $U \otimes O_R$

Assumptions
- Linearization $l \in M_{\mathbb{R}}^W$ of $R$
- $R$ quasi-symmetric, $R^{ss} = R^s$
- $\delta \in M_{\mathbb{R}}$ such that $\partial(\delta + \nabla) \cap M = \emptyset$
- $\exists v \in M_{\mathbb{R}}$ not parallel to any face of $\Sigma$

Theorem [HLS]
- Open embedding $R^s/G \subset R/G$ by restriction induces equivalence $\mathcal{M}(\delta + \nabla) \to D^b(Coh R^s/G)$.

Proof
- Fully faithfulness: “magic windows” [HL].
- Essential surjectivity: [SVdB] algorithm.

New proof [M] of Theorem [K]

GIT data
- $G = \text{GL}_k(\mathbb{K}) = \text{GL } U$
- $l = \det^{-1}$ linearization
- $V = \text{Mat}_{k \times n}(\mathbb{K})$
- $R = V \oplus V^*$
- $\mathcal{V} = (V \oplus V^*)/G$ vector bundle over $X = \text{Gr}(k, n) = V^*/G$

Representations
- $\Sigma = [-n, n]^k$
- $M_{\mathbb{R}}^+ = \{(x_1, \ldots, x_k) \mid x_1 > \cdots > x_k\}$
- $\nabla = [-\frac{n+k+1}{2}, \frac{n-k+1}{2}]^k$
- $\delta = \frac{n-k+1}{2} - 0.1$
- $(\delta + \nabla) \cap M = \{0, \ldots, n-k\}^k$
- Representations with characters in $\delta + \nabla$ correspond to Young diagrams $\lambda$ fitting in rectangle $k \times (n-k)$

Lemma: zero section
- A set of objects $S \subset D^b(\text{Coh } \mathcal{V})$ generates. $z : X \to Y$ zero section.
- Then $z^*S \subset D^b(\text{Coh } X)$ generates.

Proof
- By [HLS], $S^1U \otimes O_Y$ generate $D^b(\text{Coh } Y)$
- By lemma, $S^1U \otimes O_X \cong S^1U$ generate $D^b(\text{Coh } X)$

Bibliography