Fullness on Grassmannians via Combinatorial Constructions

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Fullness on grassmannians

Notation

- $\mathbb{K} = \overline{\mathbb{K}}$, $\operatorname{char} \mathbb{K} = 0$
- $X = \operatorname{Gr}(k, n)$

Theorem [K]

$\mathrm{D^b}(\operatorname{Coh} X) = \langle \mathrm{S}^\lambda \mathcal{U} \rangle$, where:

- $\mathcal{U}$ tautological bundle,
- λ all Young diagrams that fit in rectangle $k \times (n-k)$.

Proof [K]

- $\mathcal{K} \cong \mathcal{O}_\Delta \in \mathrm{D^b}(\operatorname{Coh} X \times X)$ Koszul complex
- $\mathcal{F} \cong p_{2*}(\mathcal{K} \otimes p_1^* \mathcal{F}) \in \mathrm{D^b}(\operatorname{Coh} X)$ written in terms of Schur functors of $\mathcal{U}$

Combinatorial constructions [HLS]

Notation

- Reductive group G
- Maximal torus $T \subset G$
- Weyl group W
- $\rho = \frac{1}{2} \sum$ (positive roots)
- G-representation R
- β_i are T-weights of R
- $M = \operatorname{Hom}(T, \mathbb{G}_m)$ character lattice
- Dominant chamber $M^+_{\mathbb{R}}$

R quasi-symmetric

For any line $L \subset M$, the sum of $\beta_i \in L$ must be 0

Zonotope $\overline{\Sigma} \subset M_{\mathbb{R}}$

$\overline{\Sigma}$ = Minkowski sum of $[0, \beta_i]$

Polytope $\nabla \subset M_{\mathbb{R}}$

(Roughly) defined by properties:

- W-invariant
- $M^+_{\mathbb{R}} \cap \nabla = M^+_{\mathbb{R}} \cap \left(-\rho + \frac{1}{2}\overline{\Sigma}\right)$

Category $\mathcal{M}(\delta + \nabla) \subset \mathrm{D^b}(R/G)$

- $\delta \in M_{\mathbb{R}}$
- $\mathrm{D^b}(\operatorname{Coh} R/G) \cong \mathrm{D^b}_G(\operatorname{Coh} R)$
- Character of $U \in \operatorname{Rep} G$ is contained in $\delta + \nabla$
- $\mathcal{M}(\delta + \nabla)$ is split generated by these $U \otimes \mathcal{O}_R$

Assumptions

- Linearization $l \in M^W_{\mathbb{R}}$ of R
- R quasi-symmetric, $R^{ss} = R^s$
- $\delta \in M^W_{\mathbb{R}}$ such that $\partial(\delta + \nabla) \cap M = \varnothing$
- $\exists v \in M^W_{\mathbb{R}}$ not parallel to any face of $\overline{\Sigma}$

Theorem [HLS]

Open embedding $R^s/G \subset R/G$ by restriction induces equivalence $\mathcal{M}(\delta + \nabla) \to \mathrm{D^b}(\operatorname{Coh} R^s/G)$.

Proof

Fully faithfulness: "magic windows" [HL].
Essential surjectivity: [ŠVdB] algorithm.

New proof [M] of Theorem [K]

GIT data

- $G = \mathrm{GL}_k(\mathbb{K}) = \mathrm{GL}\, U$
- $l = \det^{-1}$ linearization
- $V = \operatorname{Mat}_{k \times n}(\mathbb{K})$
- $R = V \oplus V^\vee$
- $\mathcal{V} = (V \oplus V^\vee)^s/G$ vector bundle over $X = \operatorname{Gr}(k, n) = V^s/G$

Representations

- $\overline{\Sigma} = [-n, n]^k$
- $M^+_{\mathbb{R}} = \{(x_1, \ldots, x_k) \mid x_1 > \cdots > x_k\}$
- $\nabla = \left[-\frac{n-k+1}{2}, \frac{n-k+1}{2}\right]^k$
- $\delta = \frac{n-k+1}{2} - 0.1$
- $(\delta + \nabla) \cap M = \{0, \ldots, n-k\}^k$
- Representations with characters in $\delta + \nabla$ correspond to Young diagrams λ fitting in rectangle $k \times (n-k)$

Lemma: zero section

A set of objects $\mathcal{S} \subset \mathrm{D^b}(\operatorname{Coh} \mathcal{V})$ generates.
$z : X \to \mathcal{V}$ zero section.
Then $z^* \mathcal{S} \subset \mathrm{D^b}(\operatorname{Coh} X)$ generates.

Proof

- By [HLS], $S^\lambda U \otimes \mathcal{O}_{\mathcal{V}}$ generate $\mathrm{D^b}(\operatorname{Coh} \mathcal{V})$
- By lemma, $S^\lambda U \otimes \mathcal{O}_X \cong S^\lambda \mathcal{U}$ generate $\mathrm{D^b}(\operatorname{Coh} X)$

Bibliography

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