Recitation 10. May 7

Focus: positive definite matrices, Markov matrices.

Definition. A symmetric matrix S is called *positive definite* if all of its eigenvalues are positive. It is *positive semidefinite* all of its eigenvalues are nonnegative, that is we allow zeroes.

Definition. A matrix A is called a *Markov matrix* if all of its entries are nonnegative and the elements in each column sum up to one. It is called a *positive Markov matrix* if in addition we require all matrix entries to be positive.

Fact. A Markov matrix A always has an eigenvalue equal to one, because columns of the matrix $A - I$ lie in the hyperplane $x_1 + \cdots + x_n = 0$. A nonpositive Markov matrix can have more than one largest eigenvalue, take for example I.

Definition. A *steady state* of a positive Markov matrix A is the unique vector v which is an eigenvector of A with eigenvalue one and whose coordinates sum up to one. It is called "steady vector", because any positive vector x whose coordinates sum up to one converges to v as we iteratively apply A, that is $\lim_n A^n x = v$.

1. Let S be a positive definite matrix. Show that then for any nonzero vector v, we have $v^T S v > 0$.

   Solution:

2. *(Strang, problem 6.5.30.)* The graph of $z = x^2 + y^2$ is a bowl opening upward, or *convex*. The graph of $z = -x^2 - y^2$ is a downward bowl, which means that it is *concave*. The graph of $z = x^2 - y^2$ is a saddle. What is a condition on a, b, c for $z = F(x, y) = ax^2 + 2bxy + cy^2$ to have a saddle point at $(0, 0)$?

   Solution:

3. If A and B are two Markov matrices, then show that their product AB is Markov as well. Further, derive that then any power A^k, $k > 0$, of a Markov matrix is Markov.

   Solution:

4. *Weather prediciton. (Inspired by en.wikipedia.org/wiki/Examples_of_Markov_chains.)*

Last May in Boston, there were 10 rainy days and 21 days without precipitation. There were 3 occasions where a rainy day followed a rainy day, 7 occasions where a dry day followed a rainy day. After a dry day, a rainy day followed on 7 occasions and another dry day happened on 13 occasions. (Note than since there are 31 days in May, there are 30 pairs of consecutive days.)

a) With the first coordinate corresponding to a rainy day and the second – to a dry day, write the vector a_1 of probabilities for what happens after a rainy day and the vector a_2 – for after a dry day.

b) Using the results from part (a), write the Markov matrix A corresponding to this setup.

c) Find a steady vector v for A.

d) Normalize v so that the sum of its coordinates equals to 1 – this will be the steady state.

e) Compare probability of having a rainy day in May 2018 with the first coordinate in the steady state vector.

Solution:

5. If a Markov matrix A has the steady state $(1, \ldots, 1)^T$, then what can you say about the rows of this matrix?

Solution: